University of Bath
Department of Economics and International Development
EC10003 Introduction to Statistics
Fall 2006
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Answers to Problems for Class 4

TRUE or FALSE problems

1. TRUE

Since $V(Z) = b^2 V(X) = \frac{1}{\sigma_X^2} \sigma_X^2 = 1$

2. TRUE

$$Var(R) = n\frac{S}{N} \cdot \frac{N-S}{N} \cdot \frac{N-n}{N-1} = \frac{96}{225}$$

$$Var(Z) = 3^2 Var(R) = 3.84$$

3. FALSE
0.75

4. FALSE
Y~N(15,54)

5. FALSE
$V(Y) = (1.5)^2 7 = 15.75$

Exercises

NCT 5.36
Cdf of binomial with n = 5 and p = 0.250000

x	P(X <= x)
0.00	0.2373
1.00	0.6328
2.00	0.8965
3.00	0.9844
4.00	0.9990
5.00	1.0000

(a) P(x ≥ 1) = 1 – Px(0) = 1 – .2373 = .7627
(b) P(x ≥ 3) = 1 – P(x = 2) = 1 - .8965 = .1035

NCT 5.44

(a) E(X) = 2000(.032) = 64

$\sigma_x = \sqrt{2000(.032)(.968)} = 7.871$

(b). Let Z = 10X

E(Z) = 10(64) = $640

σ_z = |10|(7.871) = $78.71

NCT 5.55

a. P(Shipment is accepted) can be found by: P(x = 0) with N=16, S=4, n = 4: =0.2720.

Cdf of hypergeometric with N = 16, X = 4, and n = 4

x	P(X <= x)
0.00	0.2720
1.00	0.7555
2.00	0.9731
3.00	0.9995
4.00	1.0000

b. P(Shipment is accepted) can be found by: P(x = 0) with N=16, S=1, n = 4: = 0.7500

Cdf of hypergeometric with N = 16, X = 2, and n = 4

x	P(X <= x)
0.00	0.5500
1.00	0.9500
2.00	1.0000
3.00	1.0000
4.00	1.0000

c. P(Shipment is rejected) can be found by taking 1 minus the P(Shipment is accepted): [1-P(x = 0)] with N=16, S=1, n = 4: = [1 - .75] = .25

Cdf of hypergeometric with N = 16, X = 1, and n = 4

x	P(X <= x)
0.00	0.7500
1.00	1.0000
2.00	1.0000
3.00	1.0000
4.00	1.0000

NCT 5.57

Cdf of hypergeometric with N = 12, X = 4, and n = 3

x	P(X <= x)
0.00	0.2545
1.00	0.7636
2.00	0.9818
c.	1.0000

$P(x \geq 2) = 1 - P(x \leq 1) = 1 - .7636 = .2364$

NCT 5.64

Cdf of Poisson with mu = 3.00000

x	P(X <= x)
0.00	0.0498
1.00	0.1991
2.00	0.4232
3.00	0.6472
4.00	0.8153
5.00	0.9161
6.00	0.9665
7.00	0.9881
8.00	0.9962
9.00	0.9989
c.	0.9997

$P(x \leq 2) = .4232$

NCT 5.92

Cars	P(x)	F(x)	Mean	Variance
0	0.1	0.10	0	0.48841
1	0.2	0.30	0.2	0.29282
2	0.35	0.65	0.7	0.015435
3	0.16	0.81	0.48	0.099856
4	0.12	0.93	0.48	0.384492
5	0.07	1.00	0.35	0.544887
Ex 5.57	1.00		2.21	1.8259
			S.D.	1.351259

a. E(X) = 2.21 cars sold

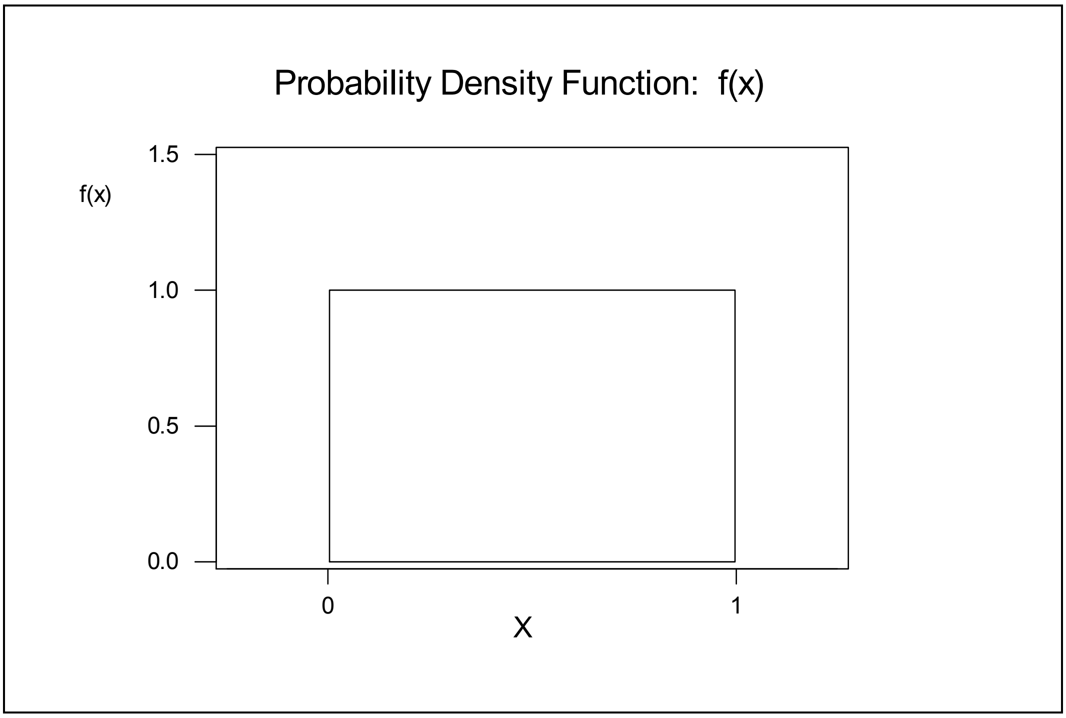
b. Standard deviation = 1.3513 cars

c. Mean Salary = \$250 + \$300 (2.21) = \$913. Standard deviation of salary = \$300(1.3513) = \$405.39

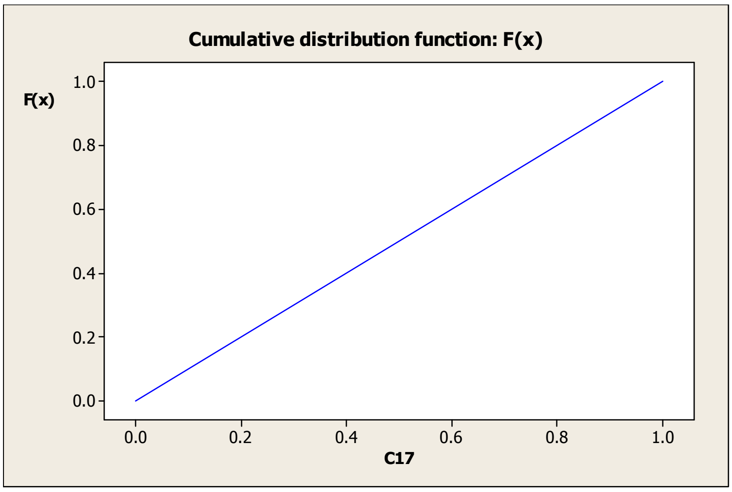
d. To earn a salary of \$1,000 or more, the salesperson must sell at least 3 cars. $P(X \geq 3) = .16 + .12 + .07 = .35$

NCT 6.5

a.



b.



c. $P(X < .25) = .25$
d. $P(X > .75) = 1 - P(X < .75) = 1 - .75 = .25$
e. $P(.2 < X < .8) = P(X < .8) – P(X < .2) = .8 - .2 = .6$

NCT 6.7

a. $P(60{,}000 < X < 72{,}000) = P(X < 72{,}000) – P(X < 60{,}000) = .6 - .5 = .1$

b. $P(X < 60{,}000) < P(X < 65{,}000) < P(X < 72{,}000)$; $.5 < P(X < 65{,}000) < .6$

NCT 6.13

$\mu_Y = 10{,}000 + 1.5\ \mu_X = 10{,}000 + 1.5\ (30{,}000) = \$55{,}000$

$\sigma_Y = |1.5|\ \sigma_X = 1.5\ (8,000) = \$12,000$

NCT 6.15
$\mu_Y = 60 + .2\ \mu_X = 60 + 140 = \$200$
$\sigma_Y = |.2|\ \sigma_X = .2\ (130) = \$26$

11. (i) If the probability of using the bus is constant and equal to 0.75 for each of the five students selected, then X follows a binomial distribution with $p = 0.75$, $n = 5$.

Then its pmf is:

$f(x) = [5!/(x!(5\text{-}x!))](0.75^x)(1\text{-}0.75)^{5-x}$
$= [5!/(x!(5\text{-}x!))](0.75^x)(0.25)^{5-x}$ for $x = 0,1,2,3,4,$

and $f(x) = 0$ otherwise

(ii) Let the cdf of X be $F(x)$

Since:

$f(0) = [5!/(0!(5\text{-}0!))](0.75^0)(1\text{-}0.75)^{5-0} = 0.25^5 = 0.00098$

$f(1) = [5!/(1!(5\text{-}1!))](0.75^1)(1\text{-}0.75)^{5-1} = (5)(0.75)(0.25^4) = 0.01465$

$f(2) = [5!/(2!(5\text{-}2!))](0.75^2)(1\text{-}0.75)^{5-2} = (5!/2!3!)(0.75^2)(0.25^3) = 0.08789$

$f(3) = [5!/(3!(5\text{-}3!))](0.75^3)(1\text{-}0.75)^{5-3} = (5!/3!2!)(0.75^3)(0.25^2) = 0.26367$

$f(4) = [5!/(4!(5\text{-}4!))](0.75^4)(1\text{-}0.75)^{5-4} = (5!/4!1!)(0.75^4)(0.25^1) = 0.39551$

$f(5) = [5!/(4!(5\text{-}4!))](0.75^5)(1\text{-}0.75)^{5-5} = (5!/5!0!)(0.75^5)(0.25^0) = 0.23730$

$$F(x) = \begin{cases} 0 & \text{for } x < 0 \\ 0.00098 & \text{for } 0 \le x < 1 \\ 0.01563 & \text{for } 1 \le x < 2 \\ 0.10352 & \text{for } 2 \le x < 3 \\ 0.36719 & \text{for } 3 \le x < 4 \\ 0.76270 & \text{for } 4 \le x < 5 \\ 1 & \text{for } 5 \le x \end{cases}$$

(iii) $P(2.5 \le X \le 3.5) = P(X \le 3.5) - P(X<2.5) = P(X \le 3.5) - P(X \le 2.5) + P(X=2.5) =$
$=0.36719 – 0.10352 + 0 = 0.26367$

12. (i) Let X be the number of aces in a hand of 5 cards. Then X follows a hypergeometric distribution with:

N=52
S=4
n=5

Then

$$P(X=3)=\frac{\frac{4!}{1!3!}\cdot\frac{48!}{46!2!}}{\frac{52!}{47!5!}}=\frac{4\cdot\frac{48\cdot 47}{2}}{\frac{52\cdot 51\cdot 50\cdot 49\cdot 48}{5\cdot 4\cdot 3\cdot 2}}=\frac{4\cdot 47\cdot 5\cdot 4\cdot 3}{52\cdot 51\cdot 50\cdot 49}=\frac{11,280}{6,497,400}=0.00174$$

(ii) Note that the probability of having exactly 3 aces did not depend on the denomination being ace, and therefore is exactly equal to the probability of drawing exactly three same cards of any denomination. Since there are 13 denominations:

P(exactly 3 cards of the same denomination)=13*0.00174=0.02262

(iii) Lex X be the number of cards of the same suit in a hand of 5 cards. Then X follows a hypergeometric distribution with

N=52
S=13
n=5

Then:

$$P(X=5)=\frac{\frac{13!}{8!5!}\cdot\frac{38!}{38!0!}}{\frac{52!}{47!5!}}=\frac{\frac{13\cdot 12\cdot 11\cdot 10\cdot 9}{5!}}{\frac{52\cdot 51\cdot 50\cdot 49\cdot 48}{5!}}=\frac{154,440}{311,875,200}=0.000495$$

Since there are four different suits:

P(all 5 cards of the same suit)=4.000495=0.00198

13.
Let X= time, in minutes, of arrival of bus

Then:

$f(x)=\frac{1}{7}$ for $10.00am \le x \le 10.07am$
$f(x)=0$ otherwise

(Since from the moment the clock strikes 10 until it reaches 10.07, the time that lapses is 7 minutes)

(a) $P(10.01am \le X \le 10.02am) = \frac{1}{10.07am - 10.00am} \cdot (10.02am - 10.01am) = \frac{1}{7}$

(b) Before 10.03 before the clocks strikes 10.03, which includes the first three minutes after 10.00 am. Hence:

$$P(X \le 10.03) = P(10.00 \le X \le 10.03) = \frac{1}{10.07 - 10.00} \cdot (10.03 - 10.00) = \frac{3}{7}$$

14.

Suppose X is discrete. Let the pmf of X be *f(X)*. We have:

$$V(X) = E\left[(X - E(X))^2\right] = E\left[X^2 - 2 \cdot X \cdot E(X) + E^2(X)\right] =$$
$$= \sum_i \left[x_i^2 - 2 \cdot x_i \cdot E(X) + E^2(X)\right] f(x_i) =$$

(following the definition of the expectation of a function of a random variable)

$$= \sum_i x_i^2 f(x_i) + \sum_i (-2 \cdot x_i \cdot E(X)) f(x_i) + \sum_i E^2(X) f(x_i) =$$
$$= E(X^2) - 2E(X) \sum_i x_i f(x_i) + E^2(X) \sum_i f(x_i) =$$

(following again the definition of the expectation of a function of a random variable for the first term, and the fact that *E(X)* and *E(X)*^{2} are constants in the second and third terms)

$$= E(X^2) - 2E(X)E(X) + E^2(X) =$$

(following the definition of the expected value of a random variable and the definition of the pmf)

$$= E(X^2) - 2E^2(X) + E^2(X) = E(X^2) - E^2(X)$$

Suppose X is continuous. Let the pdf of X be *f(X)*. We have:

$$V(X) = E\left[(X - E(X))^2\right] = E\left[X^2 - 2 \cdot X \cdot E(X) + E^2(X)\right] =$$
$$= \int_{-\infty}^{+\infty} \left[x_i^2 - 2 \cdot x_i \cdot E(X) + E^2(X)\right] f(x_i) dx =$$
$$= \int_{-\infty}^{+\infty} x_i^2 f(x_i) dx + \int_{-\infty}^{+\infty} (-2 \cdot x_i \cdot E(X)) f(x_i) dx + \int_{-\infty}^{+\infty} E^2(X) f(x_i) dx =$$
$$= E(X^2) - 2E(X) \int_{-\infty}^{+\infty} f(x_i) dx + E^2(X) \int_{-\infty}^{+\infty} f(x_i) dx =$$
$$= E(X^2) - 2E(X)E(X) + E^2(X) =$$

$= E(X^2) - 2E^2(X) + E^2(X) = E(X^2) - E^2(X)$

where all the steps follow the same logic as above.

15.
Suppose X is discrete. Let the pmf of X be *f(X)*. Then by the expectation of a function of a random variable rule:

$$E(Y) = \sum_i y_i \cdot f(x_i) = \sum_i (a + bx_i) \cdot f(x_i) = \sum_i [a \cdot f(x_i) + bx_i \cdot f(x_i)] =$$
$$= \sum_i a \cdot f(x_i) + \sum_i bx_i \cdot f(x_i) = a\sum_i f(x_i) + b\sum_i x_i \cdot f(x_i) = a + bE(X)$$

Where
$a\sum_i f(x_i) = a$ follows from the fact that $\sum_i f(x_i) = 1$
and
$b\sum_i x_i \cdot f(x_i) = bE(X)$ follows from the definition of expected value for discrete random variables.

Suppose X is continuous. Let the pdf of X be *f(X)*. Then by the expectation of a function of a random variable rule:

$$E(Y) = \int_{-\infty}^{+\infty} y_i \cdot f(x_i)dx = \int_{-\infty}^{+\infty} (a + bx_i) \cdot f(x_i)dx = \int_{-\infty}^{+\infty} [a \cdot f(x_i) + bx_i \cdot f(x_i)]dx =$$
$$= \int_{-\infty}^{+\infty} a \cdot f(x_i)dx + \int_{-\infty}^{+\infty} bx_i \cdot f(x_i)dx = a\int_{-\infty}^{+\infty} f(x_i)dx + b\int_{-\infty}^{+\infty} x_i \cdot f(x_i)dx =$$
$$= a + bE(X)$$

Where the last step follows from the definition of a pdf and the definition of the expected value of continuous random variable.

Suppose X is discrete. Let the pmf of X be *f(X)*. Then using the result of Exercise 14:

$$V(Y) = E(Y^2) - E^2(Y) = E\left[(a + bX)^2\right] - E^2(a + bX) =$$
$$= E(a^2 + 2abX + b^2X^2) - [a + bE(X)]^2 =$$
$$= \sum_i a^2 f(x_i) + \sum_i 2abx_i f(x_i) + \sum_i b^2 x_i^2 f(x_i) - \left[a^2 + 2abE(X) + b^2E^2(X)\right] =$$

(following the definition of the expectation of a function of a random variable for the first three terms, and using the result from the first part of this exercise for the last term)

$$= a^2 \sum_i f(x_i) + 2ab \sum_i x_i f(x_i) + b^2 \sum_i x_i^2 f(x_i) - a^2 - 2abE(X) - b^2 E^2(X) =$$

$$= a^2 + 2abE(X) + b^2 E(X^2) - a^2 - 2abE(X) - b^2 E^2(X) =$$

(using the definition of the expected value of a r.v. and function of a r.v.)

$$= b^2 E(X^2) - b^2 E^2(X) = b^2 \left[E(X^2) - E^2(X) \right] = b^2 V(X)$$

(using the result of Exercise 14)

For the case of X continuous follow the same logic.